Introduction to PDEs and Numerical Methods
Lecture 10.
Weighted residual methods -
Galerkin method, finite element method

Dr. Noemi Friedman, 13.01.2016.
Recap
Linear systems: strong form, weak form, minimization problem

If $A$ is symmetric positive definite

**strong form:**

\[ Ax = b \]

$\mathbf{x} \in \mathbb{R}^n$

$A \in \mathbb{R}^{n \times n}$

$b \in \mathbb{R}^n$

**weak form:**

\[ \langle Ax, y \rangle = \langle b, y \rangle \ \forall \ y \in \mathbb{R}^n \]

**minimization:**

\[ \phi(x) = \frac{1}{2} x^T Ax - x^T b \]

- Direct solvers
  - (Gauß elimination, LU/chol decomposition)
- Iterative methods
  - (Jacobi, Gauß-Seidel...)

• CG residual is orthogonal w.r.t. the energy norm to the approximating Krylov subspace

• Steepest descent
  - CG (search in the Krylov-subspace, directions are conjugates-orthogonal with respect to the energy norm)
Recap: FROM STRONG FORM TO WEAK FORM

Similarly

solving PDE $\leftrightarrow$ optimization of quadratic function

Instead of solving $Lu = f$ (strong form) $Lu = f$, $u \in D_L$, $f \in H \Rightarrow u_0 \in D_L$, $Lu_0 = f$

minimize the quadratic function:

$$F(u) = \frac{1}{2} \langle Lu, u \rangle - \langle f, u \rangle \quad \text{weak form}$$

This quadratic functional attains its **stationary point** precisely where $Lu = f$, if $L$ is symm (self-adjoint).

$$\langle Lu, v \rangle = \langle Lv, u \rangle$$

**minimum point** if $L$ is pos. def. $\langle Lu, u \rangle \geq 0$

and only zero for $u=0$
Recap: FROM STRONG FORM TO WEAK FORM
1D: Steps of formulating the weak form (recipe)

\[ Lu(x) = f(x) \]

1.) Multiply by test/weight function \( v(x) \) and integrate

\[ \langle Lu, v \rangle - \langle p, v \rangle = 0 \quad \forall v \in V \]

\[ \int Lu(x)v(x)dx - \int f(x)v(x)dx = 0 \quad \forall v \in V \]

2.) Reduce order of \( \langle Lu, v \rangle \) by using the integration by parts

3.) Apply boundary conditions

Check whether the PDE holds in the \( v(x) \) weighted average sense over \( \Omega \)
if it holds for all test functions then the PDE must hold
Recap: FROM STRONG FORM TO WEAK FORM
2D: Steps of formulating the weak form (recipe)

\[ Lu(x) = f(x) \]

1.) Multiply by test function \( \varphi \) and integrate

\[ \langle Lu, v \rangle - \langle p, v \rangle = 0 \quad \forall v \in V \]

\[ \int Lu(x)v(x)dx - \int f(x)v(x)dx = 0 \]

2.) Reduce order of \( \langle Lu, \varphi \rangle \) by using Green’s theorem (generalized integration by parts)

\[ \int_{\Omega} v \Delta u d\Omega = - \int_{\Omega} \nabla v \cdot \nabla u d\Omega + \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\Gamma \]

3.) Apply boundary conditions
Green's identity  similar to integration by part in multiple dimensions

1) rewrite equation with the product rule in multiple dimensions

\[ \nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \Delta u \]

2) integrate both sides over the domain \( \Omega \) (bounded by \( \partial \Omega \))

\[ \int_{\Omega} \nabla \cdot (v \nabla u) \, d\Omega = \int_{\Omega} \nabla v \cdot \nabla u \, d\Omega + \int_{\Omega} v \Delta u \, d\Omega \]

3) apply divergence theorem

\[ \int_{\Omega} \nabla \cdot (v \nabla u) \, d\Omega = \int_{\Omega} \text{div}(v \nabla u) \, d\Omega = \int_{\partial \Omega} (v \nabla u) \cdot n \, d\partial \Omega \]

\[ -\int_{\Omega} v \Delta u \, d\Omega = \int_{\Omega} \nabla v \cdot \nabla u \, d\Omega - \int_{\partial \Omega} (v \nabla u) \cdot n \, d\partial \Omega \]
Recap: Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.

Strong form: \[ Lu(x) = f(x) \]

Example:

\[
\begin{cases}
-\Delta u(x) = f & \text{in } \Omega, \\
u = g & \text{on } \Gamma_D, \\
\frac{\partial u}{\partial n} = h & \text{on } \Gamma_N,
\end{cases}
\]

1.) Multiply by test function \( v \) and integrate

\[
\int -\Delta u(x)v(x)d\Omega - \int f(x)v(x)d\Omega = 0
\]

2.) Reduce order of \( \langle Lu, v \rangle \) by using divergence theorem

\[
\int -\Delta u(x)v(x)d\Omega = \int_{\Omega} \nabla u(x) \cdot \nabla v(x)d\Omega - \int_{\partial \Omega} \frac{\partial u}{\partial n} v(x)d\Gamma
\]

Convert to homogeneous problem:

\[
u = \omega + \hat{u} \\
\omega: \text{known function}, \omega = g \text{ on } \Gamma_D \\
\hat{u}: \text{new function that we look for}
\]

\[
V = \{ v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \Gamma_D \}
\]
Recap: Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.

\[ \int -\Delta u(x)v(x)d\Omega = \int \nabla u(x) \cdot \nabla v(x)d\Omega - \int_{\partial \Omega} \frac{\partial u}{\partial n} v(x)d\Gamma \]

3.) Apply boundary conditions

\[ \int_{\partial \Omega} \frac{\partial u}{\partial n} v(x)d\Gamma = \int_{\Gamma_N} \frac{\partial u}{\partial n} v(x)d\Gamma + \int_{\Gamma_D} \frac{\partial u}{\partial n} v(x)d\Gamma = \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \nabla v(x) \cdot \nabla u(x)d\Omega = \int f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \nabla (\omega(x) + \hat{u}(x)) \cdot \nabla v(x)d\Omega = \int f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \nabla \hat{u}(x) \cdot \nabla v(x)d\Omega = \int f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma - \int \nabla \omega(x) \cdot \nabla v(x)d\Omega \]

from natural/Neumann BC  from essential/Dirichlet BC
Recap:
Existence and uniqueness of the solution of BVPs

<table>
<thead>
<tr>
<th>Strong form:</th>
<th>Weak form:</th>
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<tbody>
<tr>
<td>$Lu(x) = f(x)$</td>
<td>$\langle Lu, v \rangle = \langle f, v \rangle$</td>
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$\forall v \in V$  
$u \in V$

In accordance to the Lax-Milgram Lemma if:

- $F(\cdot)$ bounded, linear functional
- $a(\cdot, \cdot)$ bounded, $V$-elliptic bilinear functional and
- $V$ a Hilbert space

- solution $u$ exists
- unique solution $u \in V$
- solution $u$ depends continuously on $f$

For a specific BVP one has to find the right Hilbert space (Sobolev space or L2 space) where the conditions for $a(\cdot, \cdot)$ and $F(\cdot)$ are satisfied, and then we know, that in that space we have a unique solution!
Existence and uniqueness of the solution of BVPs — examples

1. Poisson equation:

\[-\Delta u(x) = f(x)\quad \Rightarrow\quad F(v) = \int f(x)v(x)\,dx\]

\[a(\cdot,\cdot) = \int \nabla u(x)\nabla v(x)\,dx\]

\[F(\cdot)\] linear functional, in \(H_1^0\): bounded

\[a(\cdot,\cdot)\] bilinear functional, in \(H_1^0\): bounded, V-elliptic

\[\Rightarrow\quad \begin{aligned}
\bullet & \quad \text{unique solution } u \in H_1^0 \\
\bullet & \quad v \in H_1^0
\end{aligned}\]

\[a(\cdot,\cdot)\] bilinear functional, in \(L_2\): not bounded

\[a(\cdot,\cdot)\] bilinear functional, in \(H_1\): not V-elliptic

we have to narrow down the space in which we look for the solution, because we can not prove that there is a unique solution in \(L_2\) or in \(H_1\).

2. Plate equation

\[-\Delta \Delta u(x) = f(x)\quad \Rightarrow\quad F(v) = \int f(x)v(x)\,dx\]

\[a(\cdot,\cdot) = \int \Delta u(x)\Delta v(x)\,dx\]

\[F(\cdot)\] linear functional, in \(H_2^E\): bounded

\[a(\cdot,\cdot)\] bilinear functional, in \(H_2^E\): bounded, V-elliptic

\[\Rightarrow\quad \begin{aligned}
\bullet & \quad \text{unique solution } u \in H_2^E \\
\bullet & \quad v \in H_2^E
\end{aligned}\]
Discretisation

Further simplifications (discretize to finite dimensional space)

• Approximate the solution with some basis/shape functions:
  \[ u(x) = \sum_i u_i \Phi_i(x) \]

• Instead of solving it for all \( v(x) \in V \), select finite subspace for the weighting functions:
  \[ v(x) = \sum_i u_i \varphi_i(x) \]

How to choose the subspace? How to choose the weighting functions \( v(x) \)?

• True solution can be well approximated by an element of the subspace
• Efficient computation

Bubnov-Galerkin method \((\Phi_i = \varphi_i)\)

FEM: Galerkin method with subspace of *piecewise polynomial functions*

Petrov-Galerkin method \((\Phi_i \neq \varphi_i)\)

Pointwise collocation \( \varphi_i = \delta(x - x_i) \)
Subdomain collocation \( \varphi_i = \chi_{\Omega_i} \)
Proxi model – projection of the solution

Let’s say we know that there is a unique solution in $V = H_1^0$. But $V$ is an infinite dimensional space.

Let’s narrow down the space, where we are trying to find the solution, to a finite dimensional space.

**example:**

instead of finding $u(x) \in H_1^0$
we try to find the coefficients $\alpha_j$ of a „proxi model” (ansatz function):

$$u_h (x) = \sum_{j=1}^{n} \alpha_j \omega_j (x)$$

where

$\omega_j (x)$: known (linearly independent) basis or ansatz functions

$u_h (x)$: the approximation of the solution $u(x)$, which is in an $n$-dimensional space:

$$u_h \in V_h = \text{span}\{\omega_1, \omega_2, ... \omega_n\}$$
Proxi model – projection of the solution

Let’s fix this subspace to a specific $V_h$ (that is, we fix the ansatz functions in our examples)

How do we get the best approximation of the solution in this space from the equation:

$$Lu(x) = f(x)$$

Our goal is to minimize the difference in between the solution and the approximation:

$$\text{error} = \|u(x) - u_h(x)\| < \|u(x) - z(x)\| \quad \forall z(x) \in V_h$$

The best approximation $u_h$ to $u$ from $V_h$ is the one where the error is orthogonal to the space of $V_h$, that is to all possible $z(x) \in V_h$.

Instead of writing it for all $z$ (as $z$ is an $n$-dimensional space) we can write $\forall \omega_i i = 1..n$

$$\langle (u(x) - u_h(x)), \omega_i(x) \rangle = 0 \quad i = 1..n$$
Proxi model – projection of the solution

Plugging in the proxi model to the orthogonality condition we have:

\[
\left( u - \sum_{j=1}^{n} \alpha_j \omega_j \right), \omega_i \right) = 0 \quad i = 1..n
\]

Rearranging the equation we get:

\[
\langle u, \omega_i \rangle - \sum_{j=1}^{n} \alpha_j \langle \omega_j, \omega_i \rangle = 0 \quad i = 1..n
\]

\[
\sum_{j=1}^{n} \alpha_j \langle \omega_j, \omega_i \rangle = \langle u, \omega_i \rangle \quad i = 1..n \quad \Rightarrow \quad \sum_{j=1}^{n} \alpha_j K_{ij} = f_j \quad H_2^E \quad K\alpha = f
\]
Proxi model – projection of the solution

\[ \mathbf{K} \alpha = \mathbf{f} \]

where

\[ K_{ij} = \langle \omega_j(x), \omega_i(x) \rangle \]: can be calculated from the basis functions and the inner product of the given space (if the basis is orthonormal, the matrix is the identity matrix)

\[ f_j = \langle u, \omega_i \rangle \]: ?

\[ \alpha_j \]: The coefficients that we are looking for

But how do we get \( b_j = \langle u, \omega_i \rangle \)?

We know

\[ \langle Lu, v \rangle_{L^2} = \langle f, v \rangle_{L^2} \]

If the left hand side of the weak equation is a \( V \)-elliptic, bounded bilinear functional, that is also symmetric, then it can be written as:

\[ \langle Lu, v \rangle_{L^2} = a(u, v) = \langle u, v \rangle_E \quad \rightarrow \quad f_j = \langle u, \omega_i \rangle_E = \langle f, \omega \rangle_{L^2} \]
Proxi model – projection of the solution

That means that with Galerkin method we orthogonalize the projection, that is, we minimize the error in the energy space.

And the matrix equation, that we can calculate the coefficients from, will have the form:

\[
K\alpha = f
\]

with

\[
K_{ij} = \langle \omega_j, \omega_i \rangle_E = a(\omega_j, \omega_i)
\]

\[
f_j = \langle u, \omega_i \rangle_E = \langle f, \omega_i \rangle_{L^2}
\]

Example: \(-EA \frac{d^2u}{dx^2} = p(x)\) \hspace{1cm} x = [0, l], \; u(0) = 0, \; u(l) = 0

\[
Lu = -EA \frac{\partial^2 u(x)}{\partial^2 x} \quad f = p
\]

\[
K_{ij} = \langle \omega_j(x), \omega_i(x) \rangle_E = \int_0^l EA \frac{d\omega_j(x)}{dx} \frac{d\omega_i(x)}{dx} dx \quad f_j = \langle f, \omega_i \rangle_{L^2} = \int_0^l p(x)\omega_i(x) dx
\]
The same differently: 
**Galerkin method is an orthogonal projection**

The solution of the BVP satisfies the initial weak formulation:

\[ a(u, v) = F(v) \forall v \in V \]

and as \( V_h \subset V \), it also satisfies

\[ a(u, v) = F(v) \forall v \in V_h \]

Similarly, the approximation of the solution \( u_h \) satisfies:

\[ a(u_h, v) = F(v) \forall v \in V_h \]

Subtracting the last two equations:

\[ a(u, v) - a(u_h, v) = 0 \quad \forall v \in V_h \]

Galerkin method gives the best approximation of the true solution in the given subspace \( V_h \) in the energy norm.

In case \( a(u, u) \) is a symmetric bilinear term.
Céa’s theorem

Conclusion from before:
\( a(\cdot, \cdot) \): symmetric
Gallerkin method gives the best approximation in the energy norm

But what about the error in other norms?
According to Céa’s theorem, even without \( a(\cdot, \cdot) \) being symmetric, the error of the approximation of Galerkin will be allways bounded:

\[
\| u - u_h \| \leq \frac{M}{\delta} \| u - v \| \quad \forall v \in V_h
\]

Where \( M \) and \( \delta \) are constants from the conditions of boundedness and V-ellipticity of the bilinear term \( a(\cdot, \cdot) \):
\[
a(u, v) \leq M \| u \| \| v \|
\]
\[
a(u, u) \geq \delta \| u \|^2
\]
and \( \| u - v \| \) is the norm of the difference in between the true solution and any \( v \in V_h \).
This term depends on the n-dimensional space \( V_h \) chosen, and the space where the true solution lies in.
Discretisation

How to choose \((\Phi_i)\)?

- True solution can be well approximated by an element of the subspace
- Efficient and robust computation of \(K\alpha = f\)
  - \(K\) is sparse/diagonal
  - \(K\) is well not ill-conditioned
  - Determination of and derivations with \(\Phi_i\) is easy
  - Integration/derivative of \(\Phi_i\) is easily computable

Examples:

- Polynomials
- First \(N\) eigenfunctions of the PDE: \(L\Phi_i(x) = \lambda_i \Phi_i(x)\)
- Trigonometric functions
- Piecewise polynomials → **Finite Element Method**
Discretisation
Galerkin method – choosing basis/weighting function

How about using monomials for $\Phi_i(x)$:

$$\Phi_1(x) = x, \Phi_2(x) = x^2, \Phi_3(x) = x^3 \ldots$$

$$\frac{\partial \Phi_1(x)}{\partial x} = 1, \frac{\partial \Phi_2(x)}{\partial x} = 2x, \frac{\partial \Phi_3(x)}{\partial x} = 3x^2 \ldots$$

Example (the Poisson equation):

$$K_{11} = \int_0^L \frac{\partial \Phi_1(x)}{\partial x} \frac{\partial \Phi_1(x)}{\partial x} \, dx = \int_0^L 1 \cdot 1 \, dx = L$$

$$K_{21} = \int_0^L \frac{\partial \Phi_2(x)}{\partial x} \frac{\partial \Phi_1(x)}{\partial x} \, dx = \int_0^L 1 \cdot 2x \, dx = L^2$$

$$K_{22} = \int_0^L 2x2xDx = 4/3L^3$$

$$K_{13} = \int_0^L 1 \cdot 3x^2 \, dx = L^3$$

$$K_{23} = \int_0^L 2x3x^2 \, dx = 6/4L^4$$

$$K_{33} = \int_0^L 3x^23x^2 \, dx = 9/5L^5$$

$$K_{nn} = \int_0^L nx^{n-1}nx^{n-1} \, dx = 9/5L^{2n-1} \quad \text{with} \quad K_{11} = L$$

By increasing the number of the basis functions (order), the higher the condition number of $K$ may get.

Monomials are not a good choice if high order is needed.
Discretisation
Galerkin method with nodal basis, piecewise linear shape functions

Lagrange/nodal basis:

\[ \Phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

Approximation will be exact in the mesh nodes

\[ u(x) \approx \sum_{i=1}^{N} u_i \Phi_i(x) \]

Piecewise linear basis functions (hat functions):

\[ \Phi_i(x) = N_i(x) = \begin{cases} \frac{x - x_{i-1}}{l} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} + x}{l} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \]
How to solve PDE with FEM with nodal basis, piecewise linear shape functions

Finite Element method with piecewise linear functions in 1D, hom DBC

1) Weak formulation of the PDE, definition of the ’energy’ inner product (the bilinear functional, \( a \)) and and the linear functional (\( F \))
\[
a(u, v) = F(v)
\]

2) Define approximating subspace by definition of a mesh (nodes 0,1,..N, with coordinates, elements) and setup the hat functions on them
\[
\Phi_i(x) = N_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{l} & x \in [x_{i-1}, x_i] \\
\frac{x_{i+1} + x}{l} & x \in [x_i, x_{i+1}] \\
0 & \text{else}
\end{cases} 
i = 1..N - 1
\]

3) Compute the elements of the stiffness matrix (Grammian) – evaluation of integrals
\[
K_{ij} = a(N_i(x), N_j(x)) = \left< N_i(x), N_j(x) \right>_E \quad i, j = 1..N - 1
\]

4) Compute the elements of the vector of the right hand side – evaluation of integrals
\[
f_i = F(N_i), \quad i = 1..N - 1
\]

5) Solve the system of equations: \( Ku = f \)
for \( u \), which gives the solution at the nodes.
The solution in between the nodes can be calculated from:
\[
u(x) \approx \sum_{i=1}^{N} u_i \ N_i(x)
\]
1D Example with linear nodal basis

\[ u(x) = ax \]

Strong form:
\[ -EA \frac{d^2 u}{dx^2} = p(x) \]

\[ u(0) = u(l) = 0 \]

Weak form:
\[ \int_0^l EA \frac{du}{dx} \frac{d\varphi}{dx} \, dx = \int_0^l p(x)\varphi(x) \, dx \]

Discretisation of the weak form:
\[ u(x) \approx \sum_{i=1}^{2} u_i \varphi_i(x) \quad \text{with the basis functions:} \]

\[ \varphi_1 = \begin{cases} 
3x/l & \text{for } x \leq l/3 \\
2 - 3x/l & \text{for } l/3 \leq x \leq 2l/3 \\
0 & \text{for } 2l/3 \leq x 
\end{cases} \]

\[ \frac{\partial \varphi_1}{\partial x} = \begin{cases} 
3/l & \text{for } x \leq l/3 \\
-3/l & \text{for } l/3 \leq x \leq 2l/3 \\
0 & \text{for } 2l/3 \leq x 
\end{cases} \]

\[ \varphi_2 = \begin{cases} 
0 & \text{for } x \leq l/3 \\
-1 + 3x/l & \text{for } l/3 \leq x \leq 2l/3 \\
3 - 3x/l & \text{for } 2l/3 \leq x 
\end{cases} \]

\[ \frac{\partial \varphi_2}{\partial x} = \begin{cases} 
0 & \text{for } x \leq l/3 \\
3/l & \text{for } l/3 \leq x \leq 2l/3 \\
-3/l & \text{for } 2l/3 \leq x 
\end{cases} \]
1D Example with linear nodal basis

\[
\sum_{i=1}^{N} u_i EA \int_{l} \frac{\partial \varphi_i(x)}{\partial x} \frac{\partial \varphi_j(x)}{\partial x} \, dx = \int_{l} f(x) \varphi_j(x) \, dx
\]

\[
K_{ij} = EA \int_{l} \frac{\partial \varphi_i(x)}{\partial x} \frac{\partial \varphi_j(x)}{\partial x} \, dx
\]

\[
K_{11} \frac{1}{EA} = \int_{0}^{l/3} \frac{\partial \varphi_1(x)}{\partial x} \frac{\partial \varphi_1(x)}{\partial x} \, dx + \int_{l/3}^{2l/3} \frac{\partial \varphi_1(x)}{\partial x} \frac{\partial \varphi_1(x)}{\partial x} \, dx + \int_{2l/3}^{l} \frac{\partial \varphi_1(x)}{\partial x} \frac{\partial \varphi_1(x)}{\partial x} \, dx
\]

\[
K_{11} \frac{1}{EA} = \int_{0}^{l/3} \frac{3}{l} \frac{3}{l} \, dx + \int_{l/3}^{2l/3} \frac{3}{l} \frac{3}{l} \, dx + \int_{2l/3}^{l} 0 \, dx
\]

\[
K_{11} = EA \left( \frac{3}{l} \frac{3}{l} + \frac{3}{l} + \frac{3}{l} + \right) = \frac{6}{l} EA
\]

\[
\varphi_1 = \begin{cases} 
\frac{3x}{l} & \text{for } x \leq l/3 \\
2 - \frac{3x}{l} & \text{for } l/3 \leq x \leq 2l/3 \\
0 & \text{for } 2l/3 \leq x
\end{cases}
\]

\[
\frac{\partial \varphi_1}{\partial x} = \begin{cases} 
\frac{3}{l} & \text{for } x \leq l/3 \\
-\frac{3}{l} & \text{for } l/3 \leq x \leq 2l/3 \\
0 & \text{for } 2l/3 \leq x
\end{cases}
\]

\[
\varphi_2 = \begin{cases} 
0 & \text{for } x \leq l/3 \\
-1 + \frac{3x}{l} & \text{for } l/3 \leq x \leq 2l/3 \\
3 - \frac{3x}{l} & \text{for } 2l/3 \leq x
\end{cases}
\]

\[
\frac{\partial \varphi_2}{\partial x} = \begin{cases} 
0 & \text{for } x \leq l/3 \\
\frac{3}{l} & \text{for } l/3 \leq x \leq 2l/3 \\
-\frac{3}{l} & \text{for } 2l/3 \leq x
\end{cases}
\]
1D Example with linear nodal basis

\[ K_{12} = \int_l EA \frac{\partial \varphi_1(x)}{\partial x} \frac{\partial \varphi_2(x)}{\partial x} \, dx \]

\[ \frac{K_{12}}{EA} = \int_0^{l/3} \frac{3}{l} 0 \, dx + \int_{l/3}^{2l/3} \frac{-3 + 3}{l} \, dx + \int_{2l/3}^l \frac{-3}{l} \, dx \]

\[ K_{11} = 0 - \frac{3}{l} + 0 = -\frac{3}{l} \]

\[ \frac{K_{11}}{EA} = 0 - \frac{3}{l} + 0 = -\frac{3}{l} \]

\[ K_{21} = EA \int_l \frac{\partial \varphi_2(x)}{\partial x} \frac{\partial \varphi_1(x)}{\partial x} \, dx = K_{12} = \frac{-3}{l} EA \]

\[ K_{22} = \int_l \frac{\partial \varphi_2(x)}{\partial x} \frac{\partial \varphi_2(x)}{\partial x} \, dx = \int_0^{l/3} 0 \cdot 0 \, dx + \int_{l/3}^{2l/3} \frac{3}{l} \, dx + \int_{2l/3}^l \frac{-3 - 3}{l} \, dx \]

\[ K_{22} = 0 + \frac{3}{l} + \frac{3}{l} = \frac{6}{l} \]

\[ \frac{K_{22}}{EA} = 0 + \frac{3}{l} + \frac{3}{l} = \frac{6}{l} \]

\[ K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \frac{3EA}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \]
1D Example with linear nodal basis

\[ f_j = \int_l p(x) \varphi_j(x) \, dx \]

\[ f_1 = \int_l p(x) \varphi_1(x) \, dx \]

\[ f_1 = \int_0^{l/3} ax \frac{3x}{l} \, dx + \int_{l/3}^{2l/3} ax(2 - \frac{3x}{l}) \, dx + \int_{2l/3}^l ax \cdot 0 \, dx \]

\[ f_1 = \frac{al^2}{27} + \left( \frac{3al^2}{9} - \frac{7al^2}{27} \right) + 0 = \frac{al^2}{9} \]

Home assignment:

Calculate \( f_2 = \int_l p(x) \varphi_2(x) \, dx \)

Solve the system of equations \( \frac{3EA}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \)

for \( u_1 \) and \( u_2 \). Draw the function \( u(x) \).
1D Example with linear nodal basis

Home assignment answer:

\[ f_j = \int_l p(x) \varphi_j(x) \, dx \]

\[ f_2 = \int_l p(x) \varphi_2(x) \, dx \]

\begin{align*}
f_2 &= \int_0^{l/3} ax \cdot 0 \, dx + \int_{l/3}^{2l/3} ax \left( -1 + \frac{3x}{l} \right) \, dx + \int_{2l/3}^l ax \left( 3 - \frac{3x}{l} \right) \, dx \\
&= 0 + \left( -\frac{3al^2}{18} + \frac{7al^2}{27} \right) + \left( \frac{15al^2}{18} - \frac{19al^2}{27} \right) = \frac{2al^2}{9}
\end{align*}

\[ f_2 = \frac{2al^2}{9} \]

Solve the system of equations

\[
\frac{3EA}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{al^2}{9} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{al^3}{81EA} \begin{bmatrix} 4 \\ 5 \end{bmatrix}
\]
Recommended literature

Script
Gockenbach: Partial Differential Equations

More advanced literature on FEM:
  Zienkievicz & Taylor: The Finite Element Method: Vol1, The Basis
  Gockenbach: Understanding and Implementing the Finite Element Method

+ literature given in the previous lecture