Introduction to PDEs and Numerical Methods
Tutorial 4.
Finite difference methods – stability, consistency, convergence

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Stability, consistency, convergence - introduction

Important definitions:

**Well-posedness** (in the sense of Hadamard)
- solution exists
- the solution is unique
- continuous dependence on the initial data
- e.g.: heat equation, Laplace-equation

**Ill-posed problems**

That are not well-posed in the sense of Hadamard
e.g.: inverse problems, like the inverse of the heat equation
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Well-posedness differently:

**Surjective**
\( \mathcal{L}: X \rightarrow Y \) is surjective, if every element \( y \) in \( Y \) has a corresponding element \( x \) in \( X \) such that \( f(x) = y \). The function \( f \) may map more than one element of \( X \) to the same element of \( Y \).

\[(\text{for all } y \in Y \text{ I can find a solution in } X)\]

**Injective (one-to-one mapping)**
every element of \( Y \) is the image of at most one element of \( X \)

**Continious dependence on the initial data**
\[ \| \mathcal{L}^{-1} \| < C \]

The inverse/solution operator is uniformly bounded
Numerical stability
Even if an operator is well-posed in the sense of Hadamard, it may suffer from numerical instability when solved with finite precision, or with errors in the data.

\[ \| L^{-1} \| \leq C \quad \text{but} \quad \left\| L_{\Delta t, h}^{-1} \right\| \leq C \]

A method is numerically instable if the round-off or truncation errors can be amplified, causing the error to grow exponentially.

Ill-conditioned
A well-posed operator may be ill-conditioned, that is a small error in the initial data can result in much larger errors in the answers. (indicated by a large condition number)
Consistency
A certain finite difference method is consistent if:

\[
\lim_{\Delta t, h \to 0} \| \mathcal{L}(u) - \mathcal{L}_{\Delta t, h}(u) \| = 0 \quad \text{(method approximates the differential equation)}
\]

where \( \mathcal{L}(u) \): original operator

\( \mathcal{L}_{\Delta t, h}(u) \): approximated operator (discretised)

For example:

\( \mathcal{L}(u) = u' \)

from the Taylor expansion

\[
u' = \frac{u(x + h) - u(x)}{h} + O(h)
\]

\( \mathcal{L}_h(u) = \frac{u(x + h) - u(x)}{h} \quad \text{(first order method)}\)

\[
\left\| u' - \frac{u(x + h) - u(x)}{h} \right\| \leq Ch \quad \lim_{h \to 0} \| \mathcal{L}(u) - \mathcal{L}_h(u) \| = 0
\]
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Convergence

A finite difference method is convergent if:

$$\lim_{\Delta t, h \to 0} \| u - u_{\Delta t,h} \| = 0$$

where $u$: analytical solution

$u_{\Delta t,h}$: approximated solution

Solution of the FD method (numerical approximation) gets closer to the exact solution of the PDE as the discretisation is made finer.

Difficult to show, but

Lax Richtmyer theorem
A consistent finite difference method for a well-posed, linear initial value problem is convergent if and only if it is stable.

Instead of analysing convergency check consistency and stability
Consistency

Check consistency

Derivatives are approximated with the help of the Taylor series (see derivation of difference operators in Tutorial 3).

1. Derivation of a consistent finite difference operator

Example: derivation of $u'(x)$ used in the Richardson scheme

\begin{align}
  u(x + h) &= u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \\
  u(x - h) &= u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3)
\end{align}

Subtracting from eq. (1) eq. (2) results in:

\[ u(x + h) - u(x - h) = 2u'(x)h + O(h^3) \]

\[ u'(x) = \frac{u(x + h) - u(x - h)}{2h} + O(h^2) \]

\[ u'_k = \frac{u_{k+1} - u_{k-1}}{2h} + O(h^2) \]
Consistency

2. Check consistency of an already defined scheme

Example: prove consistency of the DuFort-Frankel scheme

\[
\frac{u_{n+1,j} - u_{n-1,j}}{2\Delta t} - \frac{\beta^2}{h^2}(u_{n,j-1} - (u_{n-1,j} + u_{n+1,j}) + u_{n,j+1}) = 0
\]

\[
u_{n+1,j} = u(t + \Delta t, x) = u_{n,j} + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3)
\]

\[
u_{n-1,j} = u(t - \Delta t, x) = u_{n,j} - \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3)
\]

\[
u_{n,j+1} = u(t, x+h) = u_{n,j} + \frac{\partial u}{\partial x}h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}h^2 + O(h^3)
\]

\[
u_{n,j-1} = u(t, x-h) = u_{n,j} - \frac{\partial u}{\partial x}h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}h^2 + O(h^3)
\]

\[
\frac{\partial u}{\partial t}\frac{\Delta t + O(\Delta t^3)}{\Delta t} - \frac{\beta^2}{h^2}(2u_{n,j} + \frac{\partial^2 u}{\partial x^2}h^2 + O(h^3)) + \frac{\beta^2}{h^2}(2u_{n,j} + \frac{\partial^2 u}{\partial x^2}\Delta t^2 + O(\Delta t^3)) = 0
\]
Consistency

\[
\frac{\partial u}{\partial t} \Delta t + O(\Delta t^3) - \beta^2 \left(2u_{n,j} + \frac{\partial^2 u}{\partial x^2} h^2 + O(h^3)\right) + \beta^2 \left(2u_{n,j} + \frac{\partial^2 u}{\partial t^2} \Delta t^2 + O(\Delta t^3)\right) = 0
\]

\[
\frac{\partial u}{\partial t} - \beta^2 \frac{\partial^2 u}{\partial x^2} + E = 0
\]

\[
E = \frac{\beta^2 \Delta t^2}{h^2} \frac{\partial^2 u}{\partial t^2} + O(\Delta t^2) + O(h)
\]

The method is consistent if:
\[
\lim_{\Delta t, h \to 0} |E| = 0
\]

The second and the last term will tend to zero as discretisation if refined, but the last term will only be zero if
\[
\lim_{\Delta t, h \to 0} \left| \frac{\Delta t}{h} \right| = 0
\]

For example if the stability condition of Euler forward is satisfied:
\[
\Delta t < \frac{h^2}{2\beta^2} \quad \Rightarrow \quad \Delta t = O(h^2) \quad \Rightarrow \quad \text{scheme is consistent}
\]
Stability

Stability checking from eigenvalue analysis:

- **Method of lines**

\[ u(t) = \sum_{j=1}^{N-1} \beta_j(t)v_j \quad \beta_j(t) = \beta_j^0 e^{\lambda_j t} \]

\[ \text{eig}(A) \leq 0 \quad \Rightarrow \quad \text{unconditionally satisfied} \]

- **Euler forward method**

\[ u_{n+1} = \left( I + \Delta tA \right) u_n \quad \text{eig}(B) \leq 1 \quad \Rightarrow \quad \Delta t < \frac{h^2}{2\beta^2} \]

- **Euler backward method**

\[ u_n = \left( I - \Delta tA \right) u_{n+1} \quad \text{eig}(B_1^{-1}) \leq 1 \quad \Rightarrow \quad \text{unconditionally satisfied} \]

- **Theta method**

\[ (I - \theta \Delta tA) u_{n+1} = \left( I + (1 - \theta) \Delta tA \right) u_n \]

\[ \text{eig}(B_1^{-1}B_2) \leq 1 \quad \Rightarrow \quad \text{for } \theta \geq 1/2:\text{ unconditionally stable} \]

\[ \text{for } \theta < 1/2:\text{ } \frac{\beta^2 \Delta t}{h^2} < \frac{1}{2(1 - 2\theta)} \]
Stability

Stability checking with Von Neumann Stability Analysis

Let’s suppose our solution has the form of:

\[ u(t, x) = \sum_{m=0}^{\infty} A_m(t) e^{ikm_kx} \]  

(Fourier-expansion)

With the wave number:

\[ m = 0..M \]

Let’s suppose that the solution in time changes exponentially

\[ A_m(t) = e^{\alpha_m t} \]  

where \( \alpha_m \): constant

The solution takes the form after discretisation:

\[ t = n\Delta t, \quad x = jh \]

\[ u(n, j) = \sum_{m=0}^{M} G(k_m)^n e^{ikmjh} \]

\[ G(k_m)^n = A_m(t) = e^{\alpha_m n\Delta t} = (e^{\alpha_m \Delta t})^n \]  

gain factor/amplifier

\[ M = \frac{L}{h} \]
Stability

in simpler form

\[ u(n,j) = \sum_{k=0}^{M} G(k)^n e^{ikjh} \]

\[ u_{n,j} = G(k)^n e^{ikjh} \quad \text{for one frequency} \]

Example: let's check the stability of the following scheme for the instationary heat equation:

\[ u_{n+1,j} - u_{n,j} = \Delta t \frac{\beta^2}{h^2} (u_{n,j-1} - 2u_{n,j} + u_{n,j+1}) \quad \text{(Euler forward, three point spatial discr.)} \]

\[ G(k)^{n+1} e^{ikjh} - G(k)^n e^{ikjh} = \Delta t \frac{\beta^2}{h^2} \left( G(k)^n e^{ik(j-1)h} - 2G(k)^n e^{ikjh} + G(k)^n e^{ik(j+1)h} \right) \]

\[ G(k) e^{ikjh} - e^{ikjh} = \Delta t \frac{\beta^2}{h^2} \left( e^{ik(j-1)h} - 2e^{ikjh} + e^{ik(j+1)h} \right) \]

\[ G(k) - 1 = \Delta t \frac{\beta^2}{h^2} (e^{-ikh} - 2 + e^{ikh}) \quad e^{ikh} + e^{-ikh} = 2\cos(kh) \]

\[ G(k) = 1 + 2\Delta t \frac{\beta^2}{h^2} (\cos(kh) - 1) \]

Example: let's check the stability of the following scheme for the instationary heat equation:
Stability

The gain factor: 

$$G(k) = 1 + 2\Delta t \frac{\beta^2}{h^2} (\cos(kh) - 1)$$

In a more precise form:

$$G(k_m) = 1 + 2\Delta t \frac{\beta^2}{h^2} (\cos(k_m h) - 1)$$

$$m = 0..M \quad M = \frac{L}{h}$$

Stability requirement: 

$$|G(k_m)| \leq 1$$

$$\max(G(k_m)) : \quad m = 0 \quad G(k_0) = 1 + 2\Delta t \frac{\beta^2}{h^2} (1 - 1) = 1$$

$$\min(G(k_m)) : \quad m = M \quad G(k_M) = 1 + 2\Delta t \frac{\beta^2}{h^2} (-1 - 1) = 1 - 4\Delta t \frac{\beta^2}{h^2}$$
Stability

Stability requirement:

\[ G(k_M) = 1 - 4\Delta t \frac{\beta^2}{h^2} \geq -1 \]

\[ 4\Delta t \frac{\beta^2}{h^2} \leq 2 \]

\[ \Delta t \leq \frac{h^2}{2\beta^2} \]

Scheme for the heat equation is only stable if this condition is satisfied. (conditionally stable)

If the gain factor is positive, the solution will not oscillate in time:

\[ G(k_m) \geq 0 \]

\[ G(k_M) = 1 - 4\Delta t \frac{\beta^2}{h^2} \geq 0 \]

\[ \frac{h^2}{4\beta^2} \geq \Delta t \]

Solution will give oscillatory solution if this condition is not satisfied.