Numerical methods for PDEs
FEM - abstract formulation, the Galerkin method

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Contents of the course

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• Application to concrete formulations
• Convergence, regularity
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Abstract formulation, examples

- Introduction
- From strong form to weak form
- Weak form of BVPs with inhomogenous BCs
- Best approximation by orthogonal projection
- Orthogonal projection $\leftrightarrow$ Galerkin method, minimized error for symmetric BVPs
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Introduction to Galerkin method

Matlab code from Gockenbach book:
http://www.math.mtu.edu/~msgocken/fembook

Equations followed throughout the semester [Chapter1]:
Laplace/Poisson equation:
• bar with uniaxial load
• steady state heat flow
• small vertical deflections of a membrane
• stationary heat equation
Other elliptic BVPs:
• isotropic elasticity
FROM STRONG FORM TO WEAK FORM
1D: Steps of formulating the weak form (recipe)

\[ Lu(x) = f(x) \]

1.) Multiply by test/weight function \( v(x) \) and integrate

\[ \langle Lu, v \rangle - \langle p, v \rangle = 0 \quad \forall v \in \Omega \]

\[ \int Lu(x)v(x)dx - \int f(x)v(x)dx = 0 \quad \forall v \in \Omega \]

2.) Reduce order of \( \langle Lu, v \rangle \) by using the integration by parts

3.) Apply boundary conditions

Check whether the PDE holds in the \( v \) weighted average sense over \( \Omega \)

if it holds for all test functions from a sufficiently large set, then the PDE must hold [Gockenbach]
1D Example: bar under axial loads with inhomogeneous Neumann BC.

\[-EA \frac{d^2 u}{dx^2} = p(x)\]

I. Formulate the strong equation

1.) kinematical equation: \[\varepsilon = \frac{du}{dx}\] (strain ↔ displacement)

2.) equilibrium equation: \[\frac{d\sigma}{dx} = -\frac{p(x)}{A}\] (forces in equilibrium)

3.) constitutive equation: \[\sigma = E\varepsilon\] (stress ↔ strain)

4.) Boundary conditions: \[u(0) = 0\] Dirichlet boundary \[A\sigma(l) = t\] Neumann boundary \[u(0) = 0\] \[EA \frac{du(l)}{dx} = t\] \[x \in \Omega = [0, l]\]
1D Example: bar under axial loads with inhomogenous Neumann BC.

\[-EA \frac{d^2u}{dx^2} = p(x) \leftrightarrow \langle Lu, v \rangle - \langle p, v \rangle = 0 \quad \forall v \in H^1_0 \quad v(0) = 0 \quad EA \left. \frac{du}{dx} \right|_{x=l} = t\]

Linear term \( f(v) \):
\[
\langle p, v \rangle = \int_0^l p(x)v(x)dx
\]

Bilinear term \( a(u, v) \):
\[
\langle Lu, v \rangle = \int_0^l -EA \frac{d^2u(x)}{dx^2} v(x)dx
\]

Integration by parts:
\[
\int a'b = ab - \int ab' \quad a = -EA \frac{du}{dx} \quad b' = \frac{dv}{dx}
\]

\[
\langle Lu, v \rangle = \left[ -EA \frac{du}{dx} v \right]^l_0 - \int_0^l -EA \frac{du}{dx} \frac{dv}{dx} dx = -tv(l) + \int_0^l EA \frac{du}{dx} \frac{dv}{dx} dx
\]

\[
\langle Lu, v \rangle - \langle p, v \rangle = \int_0^l EA \frac{du}{dx} \frac{dv}{dx} dx - tv(l) - \int_0^l p(x)v(x)dx = 0
\]

\[
\int_0^l EA \frac{du}{dx} \frac{dv}{dx} dx = \int_0^l p(x)v(x)dx + tv(l) \quad \text{(principle of virtual work)}
\]
Multidimensional stationary heat equation with inhomogeneous Neumann BC.

Strong form: \[ Lu(x) = f(x) \]

Example: \[
\begin{cases}
-\Delta u(x) = f \\
u = 0 \\
\frac{\partial u}{\partial n} = h
\end{cases}
\begin{align*}
in \ \Omega, \\
on \Gamma_D, \\
on \Gamma_N,
\end{align*}
\text{u} \in C^2(\Omega)
\]

1.) Multiply by test function \( v \) and integrate
\[
\int -\Delta u(x)v(x)d\Omega - \int f(x)v(x)d\Omega = 0
\]

2.) Reduce bilinear term’s order by using Green’s identity
\[
a(u, v) = \langle Lu, v \rangle:
\begin{align*}
\int -\Delta u(x)v(x)d\Omega &= \int \nabla u(x) \cdot \nabla v(x)d\Omega - \int_{\partial \Omega} \frac{\partial u}{\partial n} v(x)d\Gamma
\end{align*}
\]
\text{V} = \{ v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \Gamma_D \}
\[
\int v\Delta u d\Omega = -\int \nabla v \cdot \nabla u d\Omega + \int_{\partial \Omega} v \frac{\partial u}{\partial n} d\Gamma
\]
Multidimensional stationary heat equation with inhomogeneous Neumann BC.

\[ \int -\Delta u(x)v(x) \, d\Omega = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, d\Omega - \int_{\partial \Omega} \frac{\partial u}{\partial n} \, v(x) \, d\Gamma \]

3.) Apply boundary conditions

\[ \int_{\partial \Omega} \frac{\partial u}{\partial n} \, v(x) \, d\Gamma = \int_{\Gamma_N} \frac{\partial u}{\partial n} \, v(x) \, d\Gamma + \int_{\Gamma_D} \frac{\partial u}{\partial n} \, v(x) \, d\Gamma = \int_{\Gamma_N} h v(x) \, d\Gamma \]

\[ \int -\Delta u(x)v(x) \, d\Omega = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, d\Omega - \int_{\Gamma_N} h v(x) \, d\Gamma \]

\[ \int \nabla u(x) \cdot \nabla v(x) \, d\Omega = \int_{\Omega} f(x)v(x) \, d\Omega + \int_{\Gamma_N} h v(x) \, d\Gamma \]
Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.

Strong form: \[ Lu(x) = f(x) \]

Example:

\[
\begin{aligned}
-\Delta u(x) &= f \\
\text{in } \Omega, & \quad u = g \\
\text{on } \Gamma_D, & \quad \frac{\partial u}{\partial n} = h \\
\text{on } \Gamma_N,
\end{aligned}
\]

1.) Multiply by test function \( v \) and integrate

\[
\int -\Delta u(x)v(x)d\Omega - \int f(x)v(x)d\Omega = 0
\]

2.) Reduce order of \( Lu \), \( v \) by using divergence theorem

\[
\int -\Delta u(x)v(x)d\Omega = \int \nabla u(x) \cdot \nabla v(x)d\Omega - \int_{\partial \Omega} \frac{\partial u}{\partial n} v(x)d\Gamma
\]

- \( \omega \): known function, \( \omega = g \) on \( \Gamma_D \)
- \( \hat{u} \): new function that we look for

\[ V = \{ v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \Gamma_D \} \]
Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.

\[ \int -\Delta u(x)v(x)d\Omega = \int \nabla u(x) \cdot \nabla v(x)d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v(x)d\Gamma \]

3.) Apply boundary conditions

\[ \int_{\partial\Omega} \frac{\partial u}{\partial n} v(x)d\Gamma = \int_{\Gamma_N} \frac{\partial u}{\partial n} v(x)d\Gamma + \int_{\Gamma_D} \frac{\partial u}{\partial n} v(x)d\Gamma = \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \int_{\Omega} \nabla u(x) \cdot \nabla v(x)d\Omega = \int_{\Omega} f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \int_{\Omega} \nabla (\omega(x) + \hat{u}(x)) \cdot \nabla v(x)d\Omega = \int_{\Omega} f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma \]

\[ \int_{\Omega} \nabla \hat{u}(x) \cdot \nabla v(x)d\Omega = \int_{\Omega} f(x)v(x)d\Omega + \int_{\Gamma_N} hv(x)d\Gamma - \int_{\Omega} \nabla \omega(x) \cdot \nabla v(x)d\Omega \]

from natural/Neumann BC             from essential/Dirichlet BC
Existence and uniqueness of the solution of BVPs

**Strong form:**

\[ Lu(x) = f(x) \]

**Weak form:**

\[ \langle Lu, v \rangle = \langle f, v \rangle \]

\[ a(u, v) \quad F(v) \]

bilinear term linearm term

**In accordance to the Lax-Milgram Lemma if:**

- \( F(\cdot) \) bounded, linear functional
- \( a(\cdot, \cdot) \) bounded, \( V \)-elliptic bilinear functional and
- \( V \) a Hilbert space

\[ \begin{align*}
\text{solution } u & \text{ exists} \\
\text{unique solution } u \in V \\
\text{solution } u \text{ depends continuously on } f
\end{align*} \]

For a specific BVP one has to find the right Hilbert space (Sobolev space or \( L^2 \) space) where the conditions for \( a(\cdot, \cdot) \) and \( F(\cdot) \) are satisfied, and then we know, that in that space we have a unique solution!
Existence and uniqueness of the solution of BVPs — examples

1. Poisson equation:
   \[-\Delta u(x) = f(x)\]
   \[F(v) = \int f(x)v(x)\,dx\]
   \[a(\cdot,\cdot) = \int \nabla u(x)\nabla v(x)\,dx\]
   \[F(\cdot) \text{ linear functional, in } H_0^1: \text{ bounded} \]
   \[a(\cdot;\cdot) \text{ bilinear functional, in } H_0^1: \text{ bounded, V-elliptic} \]
   \[\text{we have to narrow down the space in which we look for the solution, because we can not prove that there is a unique solution in } L_2 \text{ or in } H_1\]

2. Plate equation
   \[-\Delta\Delta u(x) = f(x)\]
   \[F(v) = \int f(x)v(x)\,dx\]
   \[a(\cdot;\cdot) = \int \Delta u(x)\Delta v(x)\,dx\]
   \[F(\cdot) \text{ linear functional, in } H_2^E: \text{ bounded} \]
   \[a(\cdot;\cdot) \text{ bilinear functional, in } H_2^E: \text{ bounded, V-elliptic} \]
   \[\text{unique solution } u \in H_2^E \]
   \[v \in H_2^E \]
Proxi model – projection of the solution

Let’s say we know that there is a unique solution in $V = H^0_1$.
But $V$ is an infinite dimensional space

Let’s narrow down the space, where we are trying to find the solution, to a finite dimensional space

example:

instead of finding $u(x) \in H^0_1$
we try to find the coefficients $\alpha_j$ of a „proxi model“ (ansatz function):

$$u_h (x) = \sum_{j=1}^{n} \alpha_j \omega_j (x)$$

where

$\omega_j (x)$: known (linearly independent) basis or ansatz functions
$u_h (x)$: the approximation of the solution $u(x)$, which is in an n-dimensional space:

$$u_h \in V_h = \text{span}\{\omega_1, \omega_2, ... \omega_n\}$$
Proxi model – projection of the solution

Let’s fix this subspace to a specific $V_h$ (that is, we fix the ansatz functions in our examples)

How do we get the best approximation of the solution in this space from the equation:

$$Lu(x) = f(x)$$

Our goal is to minimize the difference in between the solution and the approximation:

$$error = \|u(x) - u_h(x)\| < \|u(x) - z(x)\| \quad \forall z(x) \in V_h$$

The best approximation $u_h$ to $u$ from $V_h$ is the one where the error is orthogonal to the space of $V_h$, that is to all possible $z(x) \in V_h$.

Instead of writing it for all $z$ (as $z$ is an n-dimensional space) we can write $\forall \omega_i i = 1..n$

$$\langle (u(x) - u_h(x)), \omega_i(x) \rangle = 0 \quad i = 1..n$$
Proxi model – projection of the solution

Plugging in the proxi model to the orthogonality condition we have:

\[ \left( u - \sum_{j=1}^{n} \alpha_j \omega_j \right) \cdot \omega_i = 0 \quad i = 1..n \]

Rearranging the equation we get:

\[ \langle u, \omega_i \rangle - \sum_{j=1}^{n} \alpha_j \langle \omega_j, \omega_i \rangle = 0 \quad i = 1..n \]

\[ \sum_{j=1}^{n} \alpha_j \langle \omega_j, \omega_i \rangle = \langle u, \omega_i \rangle \quad i = 1..n \quad \Rightarrow \quad \sum_{j=1}^{n} \alpha_j G_{ij} = b_j \quad \Rightarrow \quad G\alpha = b \]
Proxi model – projection of the solution

\[ G\alpha = b \]

where

\[ G_{ij} = \langle \omega_j, \omega_i \rangle \]: can be calculated from the basis functions and the inner product of the given space (if the basis is orthonormal the matrix is the identity matrix

\[ b_j = \langle u, \omega_i \rangle \]: ?

\[ \alpha_j \]: The coefficients that we are looking for

But how do we get \( b_j = \langle u, \omega_i \rangle \)?

We know

\[ \langle Lu, v \rangle_{L^2} = \langle f, v \rangle_{L^2} \]

If the left hand side of the weak equation is a V-elliptic, bounded bilinear functional, that is also symmetric, then it can be written as:

\[ \langle Lu, v \rangle_{L^2} = a(u, v) = \langle u, v \rangle_E \]
Proxi model – projection of the solution

That means that with Galerkin method we orthogonalize the projection, that is, we minimize the error in the energy space.

And the matrix equation, that we can calculate the coefficients from, will have the form:

$$G\alpha = b$$

with

$$G_{ij} = \langle \omega_j, \omega_i \rangle_E = a(\omega_j, \omega_i)$$

$$b_j = \langle u, \omega_i \rangle_E = \langle f, \omega_i \rangle_{L^2}$$

Example: \(-EA \frac{d^2u}{dx^2} = p(x)\) \quad \begin{align*} x &= [0, l], \quad u(0) = 0, \quad u(l) = 0 \\ Lu &= -EA \frac{\partial^2 u(x)}{\partial^2 x} \quad f = p \\ G_{ij} &= \langle \omega_j(x), \omega_i(x) \rangle_E = \int_0^l EA \frac{d\omega_j(x)}{dx} \frac{d\omega_i(x)}{dx} \, dx \quad b_j = \langle f, \omega_i \rangle_{L^2} = \int_0^l p(x)\omega_i(x) \, dx \end{align*}
Galerkin method is an orthogonal projection

The solution of the BVP satisfies the initial weak formulation:

\[ a(u, v) = F(v) \quad \forall v \in V \]

and as \( V_h \subset V \), it also satisfies

\[ a(u, v) = F(v) \quad \forall v \in V_h \]

Similarly, the approximation of the solution \( u_h \) satisfies:

\[ a(u_h, v) = F(v) \quad \forall v \in V_h \]

Subtracting the last two equations:

\[ a(u, v) - a(u_h, v) = 0 \quad \forall v \in V_h \]

\[ a(u - u_h, v) = 0 \quad \forall v \in V_h \]

Galerkin method gives the best approximation of the true solution in the given subspace \( V_h \) in the energy norm.

In case \( a(u, u) \) is a symmetric bilinear term.
Céa’s theorem

Conclusion from before:
\(a(\cdot, \cdot)\): symmetric  \(\Rightarrow\) Gallerkin method gives the best approximation in the energy norm

But what about the error in other norms?
According to Céa’s theorem (see prove at the lecture note), even without \(a(\cdot, \cdot)\) being symmetric, the error of the approximation of Galerkin will be always bounded:

\[
\|u - u_h\| \leq \frac{M}{\delta} \|u - v\| \quad \forall v \in V_h
\]

Where \(M\) and \(\delta\) are constants from the conditions of boundedness and V-ellipticity of the bilinear term \(a(\cdot, \cdot)\):

\[
\begin{align*}
    a(u, v) &\leq M\|u\|\|v\| \\
    a(u, u) &\geq \delta\|u\|^2
\end{align*}
\]

and \(\|u - v\|\) is the norm of the difference in between the true solution and any \(v \in V_h\).
This term depends on the n-dimensional space \(V_h\) chosen, and the space where the true solution lies in.