Inelastic Media under Uncertainty: Stochastic Models and Computational Approaches

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Overview

1. Why stochastic models?
2. Possible computational approaches
3. Model problem and mathematical formulation
4. Discretisation of random fields
5. Stochastic Galerkin Methods
6. Computations at a material point
7. Plane strain example
Why Probabilistic or Stochastic Models?

Systems with inelastic materials contain uncertain elements, as some details are not precisely known.

- Loading of the material, e.g. due to wind, waves, etc.
- More generally, the action from the surrounding environment.
- The system itself may contain only incompletely known parameters, processes or fields (not possible or too costly to measure)
- There may be small, unresolved scales in the model, they act as a kind of background noise.

All these items introduce some uncertainty in the model.
A bit of ontology:

- **Uncertainty** may be *aleatoric*, which means random and not reducible, or

- *epistemic*, which means due to incomplete knowledge.

Stochastic models give *quantitative* information about uncertainty, they are used for both types of uncertainty.

Possible areas of use: Reliability, heterogeneous materials, upscaling, incomplete knowledge of details, uncertain [inter-]action with environment, random loading, etc.
Probability

What is probability? We may understand probability as

- A mathematical concept — theory of a finite measure.
- Applies to aleatoric phenomena, i.e. frequencies of occurrence—Bernoulli’s law of large numbers.
- Applies also to epistemic concepts — extension of Aristotelian propositional logic to uncertain propositions — Cox’s theorem. Realm of Bayesian and maximum entropy methods.

Exclusive application to first area is today often labeled classical, historically Bernoulli and Laplace had both in mind.
General Problem Description

We want to approach a model inelastic (irreversible) problem, namely linear elasticity combined with perfect plasticity under quasi-static loading, where both the reversible elastic properties and the irreversible plastic properties are random.

Thus the model system is described by partial differential equations with random coefficients.

In general, there are several mathematical/numerical methods to deal with the time evolution of such stochastic systems, they each involve very different views of the underlying randomness.
Model Problem

Quasistatic Linear Elasticity with Perfect Plasticity

Loading history $f(t) \mid [f(x, t)]$ & appr. boundary conditions

Equilibrium $\forall t : \langle D^*\sigma, v \rangle_V = \langle f, v \rangle_V := \int_G f(x, t) \cdot v(x) \, dx \quad \forall v$

and strain definition $\langle Du, \tau \rangle_\Sigma = \langle \epsilon, \tau \rangle_\Sigma := \int_G \epsilon(x, t) : \tau(x) \, dx \quad \forall \tau$

Elastic-plastic strain split $\epsilon = \epsilon_e + \epsilon_p$

Elastic constitutive model $\langle \epsilon, A\epsilon_e \rangle_\Sigma = \langle \epsilon, \sigma \rangle_\Sigma \quad \forall \epsilon$

Elastic domain $\sigma \in K$, a convex set,

and plastic flow rule $\langle \dot{\epsilon}_p, \sigma - \tau \rangle_\Sigma \leq 0 \quad \forall \tau \in K$
Heterogenous Material

- Heterogeneities at the micro-structural level are usually subject to a number of uncertainties.

- Material properties of heterogeneous material usually not known with certainty at each point.

- Uncertainty $\Rightarrow$ stochasticity.

- Heterogeneous material behaves according to an elasto-plastic model, but with uncertain parameters, which are modeled as random fields.
Stochastic Model Problem

Assume loading $f$, elastic law $A$, and elastic domain $\mathcal{K}$ (plastic flow rule) are uncertain, given by probabilistic model with realisations $\omega \in \Omega$, where $\Omega$ is a probability space with probability measure $\mathbb{P}$.

**Loading history** $f(t, \omega)$ \quad $[f(x, t, \omega)]$

**Equilibrium** $\forall t : \langle \langle D^*\sigma, v \rangle \rangle_{\mathcal{V}} = \langle \langle f, v \rangle \rangle_{\mathcal{V}} := \int_\Omega \langle f(\omega), v(\omega) \rangle_{\mathcal{V}} \mathbb{P}(d\omega)$ \quad $\forall v$

**Strain** $\langle \langle Du, \tau \rangle \rangle_{\Sigma} = \langle \langle \varepsilon, \tau \rangle \rangle_{\Sigma} := \int_\Omega \langle \varepsilon(\omega), \tau(\omega) \rangle_{\Sigma} \mathbb{P}(d\omega)$ \quad $\forall \tau$

**Elastic constitutive model** $\langle \langle \varepsilon, A\varepsilon_e \rangle \rangle_{\Sigma} = \langle \langle \varepsilon, \sigma \rangle \rangle_{\Sigma}$ \quad $\forall \varepsilon$

**Elastic domain** $\sigma(t, \omega) \in \mathcal{K}(\omega)$ \quad $\forall \omega \in \Omega$ (a.s.)

**Plastic flow rule** $\langle \langle \dot{\varepsilon}_p, \sigma - \tau \rangle \rangle_{\Sigma} \leq 0$ \quad $\forall \tau \in \mathcal{K}$
Simplified Stochastic Model Assumptions

- **Isotropic** material (unrealistic for heterogeneous realisations)

- Plasticity with **von Mises** yield criterion, (isotropy unrealistic for heterogeneous realisations)

- The **simplest** case of an **irreversible** material behaviour.

- Material behaviour is described by three **random fields**: Bulk modulus $K(x, \omega)$, shear modulus $G(x, \omega)$, and yield stress $\sigma_y(x, \omega)$, solution quantities like $u(x, t, \omega)$, $\varepsilon_e(x, t, \omega)$, and $\varepsilon_p(x, t, \omega)$ are also random fields.

- For computational experiments: **Plane strain** conditions.
Stochastic Model

- **Uncertainty** of system parameters—e.g.
  \[ K = K(x, \omega) = \bar{K}(x) + \tilde{K}(x, \omega), \]
  \[ G = G(x, \omega), \quad \sigma_y = \sigma_y(x, \omega) \text{ are stochastic fields.} \]

- These parameters have to be **positive** \( \Rightarrow \) here assumed log-normal.

- Let \( \kappa \) be one of \( K, G, \sigma_y \), we assume \( \kappa(x, \omega) \geq \kappa_0(x) \quad \forall \omega \text{ (a.s.)} \)

- Modified **log-normal** distribution
  \[ \kappa(x, \omega) = \kappa_0(x) + \kappa_1(x) \exp(\gamma(x, \omega)), \]
  where \( \gamma(x, \omega) \) is a **Gaussian** random field.
Realisation of $\kappa(x, \omega)$
General Stochastic Methods

- **Moments**: Derive equations for the moments of the quantities of interest. Usually Perturbation.
- **Probability distributions / densities**: Derive equations for the probability densities, e.g. Master-Equation, Fokker-Planck, e.g. Jeremić.
- **Direct Integration**: Compute desired statistics via direct integration (high dimensional, e.g. [quasi] Monte Carlo, Smolyak [= sparse grids]).
- **Direct Approximation**: Compute an approximation to solution fields, use this to compute everything else (traditional response surface methods, White Noise Analysis, stochastic Galerkin, stochastic collocation).
- **Approximate descriptions** for plasticity, like the theory of bounding bodies, e.g. Muneo and Hori.
References (Incomplete)

Formulation of PDEs with random coefficients, i.e. Stochastic Partial Differential Equations (SPDEs):
Babuška, Tempone, Nobile; Glimm; Holden, Øksendal; Karniadakis, Xiu; Lions; HGM, Keese; Schwab, Tudor; Zabaras

Spatial/temporal expansion of stochastic processes/ random fields:
Adler; Karhunen, Loève; Krée, Soize; Wiener

White noise analysis/ polynomial chaos (PCE)/ multiple Itô integrals:
Wiener; Cameron, Martin; Hida, Potthoff; Holden, Øksendal; Itô; Kondratiev; Malliavin

Galerkin / collocation methods for SPDEs:
Babuška, Tempone, Nobile; Benth, Gjerde; Cao; Ghanem, Spanos; Karniadakis, Xiu, Lucor; HGM, Keese; Schwab, Tudor; Zabaras
Mathematical Results for Elasticity

- For linear elasticity, problem is well-posed in the sense of Hadamard (existence, uniqueness, continuous dependence on data).
- For linear elasticity, discretisation may be achieved by stochastic Galerkin methods, convergence established with Céa’s lemma.
- Galerkin methods are stable, if no variational crimes are committed.
- Computational methods exist, efficient ones under development.
- Equations have structure of a tensor product (storage and use).

\[ K u = \sum_j \sum_{\alpha} \xi_j^{(\alpha)} \Delta^{(\alpha)} \otimes K_j u = f \]

- Good approximating stochastic subspaces have to be found.
Sparsity Structure of linear Equations

Non-zero blocks of $\Delta^{(\alpha)}$ for increasing degree of $H_\alpha$
Block-Diagonal Pre-Conditioner

Let \( \overline{K} = K_0 = \) stiffness-matrix for average material \( \overline{\kappa}(x) \).

Use deterministic solver as pre-conditioner:

\[
P = \begin{pmatrix}
\overline{K} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \overline{K}
\end{pmatrix} = I \otimes \overline{K}
\]

Good pre-conditioner, when variance of \( \kappa \) not too large. Otherwise use \( P = \text{block-diag}(K) \). This may again be done with existing deterministic solver.

Block-diagonal \( P \) is well suited for parallelisation.
Functionals of Interest

Desirable: Uncertainty Quantification or Optimisation under uncertainty:

The goal is to compute functionals of the solution:

\[ \Psi_u = \langle \Psi(u) \rangle := \mathbb{E}(\Psi(u)) := \int_{\Omega} \int_{G} \Psi(u(x, \omega), x, \omega) \, dx \, \mathbb{P}(d\omega) \]

e.g.: \( \bar{u} = \mathbb{E}(u) \), or \( \text{var}_u = \mathbb{E}((\tilde{u})^2) \), where \( \tilde{u} = u - \bar{u} \),
or \( \mathbb{P}\{u \leq u_0\} = \mathbb{P}(\{\omega \in \Omega | u(\omega) \leq u_0\}) = \mathbb{E}(\chi_{\{u \leq u_0\}}) \)

All desirables are usually expected values of some functional, to be computed via (high dimensional) integration over \( \Omega \).
Methods distinguished by how expensive evaluation of \( u(\omega_z) \) in sum, and how many points \( \omega_z \) needed.
General Computational Approach

Principal Approach:

1. **Discretise / approximate** physical model (e.g. via finite elements, finite differences), and **approximate** stochastic model (processes, fields) in **finitely many independent** random variables (RVs), ⇒ **stochastic discretisation**.

2. Compute statistics via integration over $\Omega$—**high dimensional** (e.g. Monte Carlo, Quasi Monte Carlo, Smolyak (= sparse grids)):
   - Via direct integration. Each integration point $\omega_z \in \Omega$ requires one expensive PDE solution (with rough data).
   - Or approximate solution with some **response-surface**, then integration by sampling a **cheap** expression at each integration point.
Computational Requirements

- How to represent a stochastic process for computation, both simulation or otherwise?
- Best would be as some combination of countably many independent random variables (RVs).
- How to compute the required integrals or expectations numerically?
- Best would be to have probability measure as a product measure $\mathbb{P} = \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_\ell$, then integrals can be computed as iterated one-dimensional integrals via Fubini’s theorem,

$$\int_{\Omega} \Psi \mathbb{P}(d\omega) = \int_{\Omega_1} \ldots \int_{\Omega_\ell} \Psi \mathbb{P}_1(d\omega_1) \ldots \mathbb{P}_\ell(d\omega_\ell)$$
A \( \mathcal{V} \)-valued random variable (RV) \( r \) is a map \( \Omega \rightarrow \mathcal{V} \) (mostly \( \mathcal{V} = \mathbb{R} \)) completely specified by its distribution function

\[
\forall r \in \mathbb{R} : F_r(r) := \Pr\{r(\omega) \leq r\} := \int_{\{r(\omega) \leq r\}} \mathbb{P}(d\omega) = \mathbb{E}\left(\chi_{\{r(\omega) \leq r\}}\right).
\]

Mean \( \bar{r} = \mathbb{E}(r(\cdot)) \), [auto-]covariance \( C_r := \mathbb{E}(\tilde{r} \otimes \tilde{r}) \), and fluctuating part \( \tilde{r}(\omega) = r(\omega) - \bar{r} \), with \( \mathbb{E}(\tilde{r}) = 0 \).

Two RVs \( r_1 \) and \( r_2 \) are

**uncorrelated** If the [cross-]covariance \( C_{r_1, r_2} := \mathbb{E}(\tilde{r}_1 \otimes \tilde{r}_2) = 0 \), or if in case \( \mathcal{V} = \mathbb{R} \): \( \langle \tilde{r}_1, \tilde{r}_2 \rangle := \mathbb{E}(\tilde{r}_1 \tilde{r}_2) = 0 \) (orthogonal).

**independent** if for all functions \( \phi_1 \) and \( \phi_2 \) it holds that

\[
\mathbb{E}(\phi_1(r_1)\phi_2(r_2)) \equiv \mathbb{E}(\phi_1(r_1))\mathbb{E}(\phi_2(r_2)).
\]
Random Fields

Mean \( \bar{\kappa}(x) = \mathbb{E}(\kappa_\omega(x)) \) and fluctuating part \( \tilde{\kappa}(x, \omega) \).

Covariance may be considered at different positions
\[
C_\kappa(x_1, x_2) := \mathbb{E}(\tilde{\kappa}(x_1, \cdot) \otimes \tilde{\kappa}(x_2, \cdot))
\]

If \( \bar{\kappa}(x) \equiv \bar{\kappa} \), and \( C_\kappa(x_1, x_2) = c_\kappa(x_1 - x_2) \), process is homogeneous.

Here representation through spectrum as a Fourier sum is well known.

• Need to discretise spatial aspect (generalise Fourier representation).
  One possibility is the Karhunen-Loève expansion (KLE).

• Need to discretise each of the random variables in Fourier synthesis.
  One possibility is Wiener’s polynomial chaos expansion (PCE).
**Karhunen-Loève Expansion I**

**mode 1:**

**mode 15:**

**KLE:** Other names: * Proper Orthogonal Decomposition (POD), Singular Value Decomposition (SVD), Principal Component Analysis (PCA):* spectrum of \( \{ \kappa_j^2 \} \subset \mathbb{R}_+ \) and orthogonal KLE eigenfunctions \( g_j(x) \):

\[
\int_G C_\kappa(x, y) g_j(y) \, dy = \kappa_j^2 g_j(x) \quad \text{with} \quad \int_G g_j(x) g_k(x) \, dx = \delta_{jk}.
\]

\[\Rightarrow\] Mercer’s representation of \( C_\kappa \):

\[
C_\kappa(x, y) = \sum_{j=1}^{\infty} \kappa_j^2 g_j(x) g_j(y)
\]
Karhunen-Loève Expansion II

Representation of $\kappa$:

$$\kappa(x, \omega) = \bar{\kappa}(x) + \sum_{j=1}^{\infty} \kappa_j g_j(x) \xi_j(\omega) =: \sum_{j=0}^{\infty} \kappa_j g_j(x) \xi_j(\omega)$$

with centred, normalised, uncorrelated random variables $\xi_j(\omega)$:

$$\mathbb{E}(\xi_j) = 0, \quad \mathbb{E}(\xi_j \xi_k) =: \langle \xi_j, \xi_k \rangle_{L_2(\Omega)} = \delta_{jk}.$$
Karhunen-Loève Expansion III

Realisation with:

6 modes  15 modes  40 modes

Truncate after \( m \) largest eigenvalues
⇒ optimal—in variance—expansion in \( m \) RVs.
Approximating Random Fields

The solution \( u(x, \omega) \) will be a random field through \( \xi_j(\omega) \), i.e. \( u(x, \omega) = u(x, \xi_j(\omega)) \).

How to deal with RVs \( \xi_j(\omega) \) ?

- Use \( \xi_j(\omega) \) directly. Assume \( \xi_j(\omega) \) to be independent, only a finite number \( M \). Transform measure \( \mathbb{P} \) to \( Y = \mathbb{R}^M \) with image measure from \( \{ \xi_j(\omega) \}_{j=1,...,M} \). Ansatz for solution \( u(x, \omega) \) in (doubly orthogonal) polynomials in \( y = (y_1, \ldots, y_M) \in Y \) w.r.t. image measures.

- Represent \( \xi_j(\omega) \) as functions of other—simpler—RVs.
Functions of Simpler RVs

What kind of simpler RVs?
What kind of functions? — Usually polynomials or other algebras.

- Gaussian RVs — classical Wiener Chaos (White Noise Analysis)
- Poissonian RVs — discrete Poisson Chaos
- other RVs, e.g. uniform, exponential, Gamma, Beta, etc.

This is called generalised Polynomial Chaos (gPC).

Best is to use orthogonal polynomials w.r.t. relevant measure, i.e.

Hermite polynomials for Gaussian RVs, Charlier polynomials for Poisson RVs, Legendre polynomials for uniform RVs, Laguerre polynomials for exponential RVs, etc. ⇒ Askey scheme.
Each \( \xi_j(\omega) = \sum_\alpha \xi_j^{(\alpha)} H_\alpha(\theta(\omega)) \) from KLE may be expanded in polynomial chaos expansion (PCE), with orthogonal polynomials of independent Gaussian RVs \( \{\theta_m(\omega)\}_{m=1}^{\infty} =: \theta(\omega) \):

\[
H_\alpha(\theta(\omega)) = \prod_{j=1}^{\infty} h_\alpha_j(\theta_j(\omega)),
\]

where \( h_\ell(\vartheta) \) are the usual Hermite polynomials, and

\[
\mathcal{J} := \{\alpha \mid \alpha = (\alpha_1, \ldots, \alpha_j, \ldots), \alpha_j \in \mathbb{N}_0, |\alpha| := \sum_{j=1}^{\infty} \alpha_j < \infty \}
\]

are multi-indices, where only finitely many of the \( \alpha_j \) are non-zero.

Here \( \langle H_\alpha, H_\beta \rangle_{L_2(\Omega)} = \mathbb{E}(H_\alpha H_\beta) = \alpha! \delta_{\alpha,\beta}, \) where \( \alpha! := \prod_{j=1}^{\infty} (\alpha_j!) \).
Hermite polynomials $H_\alpha(\theta)$ are considered on $\Theta = \mathbb{R}^N$ with image product measure $\Gamma = \bigotimes_m \Gamma_m$ from Gaussian RVs $\{\theta_m(\omega)\}_{m=1}^\infty =: \theta(\omega)$. Remember that polynomials are an algebra:

$$h_k(\vartheta)h_\ell(\vartheta) = \sum_{m=0}^{k+\ell} c_{k\ell}^{(m)} h_m(\vartheta), \quad c_{k\ell}^{(m)} = \frac{k! \ell!}{(g-k)! (g-\ell)! (g-m)!}$$

The coefficients—nonzero only for integer $g = (k + \ell + m)/2$ and $g \geq k, \ell, m$—are structure constants of the algebra.

Similarly for multi-polynomials $H_\alpha$:

$$H_\alpha(\theta)H_\beta(\theta) = \sum_\gamma c^{(\gamma)}_{\alpha\beta} H_\gamma(\theta)$$

Structure constants $c^{(\gamma)}_{\alpha\beta} = \prod_j c^{(\gamma_j)}_{\alpha_j\beta_j}$
Time Discrete Computation

- Increment time from $t_0$ by $\Delta t$ to $t_1 := t_0 + \Delta t$, quantities at $t_0$ are $[\cdot]_0$, those at $t_1$ are $[\cdot]_1$, and increments $\Delta[\cdot]$.
- Increment loading to $f_1 = f(t + \Delta t) = f(t) + \Delta f = f_0 + \Delta f$
- Equilibrium loop: Find $\Delta u$ such that $D^*\sigma_1 = f_1$ and $\sigma_1 \in \mathcal{K}$.
  - Predict by [tangential] elasticity a $\Delta u$ from residual $\Delta f$, giving strain increment $\Delta \varepsilon = D\Delta u$.
  - Compute trial elastic stress $\hat{\sigma}_1 = \sigma_0 + A\Delta \varepsilon$,
    project onto convex elastic domain, giving trial stress $\hat{\sigma}_1 \in \mathcal{K}$.
  - Compute trial elastic strain $\hat{\varepsilon}_{e1} = A^{-1}\hat{\sigma}_1$
    and trial plastic strain $\hat{\varepsilon}_{p1} = (\varepsilon_0 + \Delta \varepsilon) - \hat{\varepsilon}_{e1}$.
- Check equilibrium, otherwise repeat loop with residuum $\Delta f := f_1 - D^*\hat{\sigma}_1$. 
Material Point

- The elastic-plastic evolution at each point is independent from all other material points.

- Consideration of only one material point

- Consideration of one typical discrete time-step for plasticity computation

- Definition of a stochastic version of the radial return mapping algorithm
Quantities at a Material Point

All random fields $\kappa(x, \omega)$ at a material point (Gauss point in FEM) $x_0$ are simple random variables $\kappa(\omega) := \kappa(x_0, \omega)$.

Random fields have KLE / PCE approximation

$$\kappa(x, \omega) = \sum_{j=0}^{M} \kappa_j g_j(x) \xi_j(\omega) = \sum_{j=0}^{M} \kappa_j g_j(x) \sum_{\alpha \in J_M} \xi_j^{(\alpha)} H_\alpha(\omega)$$

$\Rightarrow$ PCE of point variable:

$$\kappa(\omega) = \sum_{\alpha \in J_M} \kappa^{(\alpha)} H_\alpha(\omega) = \sum_{\alpha \in J_M} (\xi_j^{(\alpha)} \sum_{j=0}^{M} \kappa_j g_j(x_0)) H_\alpha(\omega)$$
Realisation of Random Variables

![Graphs showing the realisation of random variables with different PDFs and K distributions.](image)
Stochastic Radial Return Map—Input

- lognormal random variables $K(\omega), G(\omega), \sigma_y(\omega)$
- stress at beginning of the step $\sigma_0(\omega) = -p_0(\omega)I + \sigma_0^D(\omega)$
- initial pressure $p_0(\omega)$
- initial deviatoric part of the stress $\sigma_0^D(\omega)$
- random variables $\sigma_0(\omega), p_0(\omega), \sigma_0^D(\omega)$
- realization of all involved random variables $\omega$
- strain increment $\Delta\varepsilon(\omega)$
Find:

- stress at the end of step \( \sigma_1(\omega) = -p_1(\omega)I + \sigma_1^D(\omega) \)
- elastic and plastic strain at the end of the step \( \Delta \varepsilon_e(\omega), \Delta \varepsilon_p(\omega) \)
- trial deviatoric elastic stress
  \[
  \sigma_e^D(\omega) = \sigma_0^D(\omega) + G(\omega)(\Delta \varepsilon(\omega) - \frac{1}{3} \text{tr}(\Delta \varepsilon(\omega)))
  \]
- pressure \( p_1(\omega) = p_0(\omega) - \frac{1}{3}K(\omega)\text{tr}\Delta \varepsilon(\omega) \)
• Yield criteria: require for all \( \omega \) (almost surely) that \( \sigma_1(\omega) \) is inside yield surface:

\[
\sigma_1^D(\omega) = \sigma_y(\omega) \frac{\sigma_e^D(\omega)}{\|\sigma_e^D(\omega)\| \ J_2}
\]

• elastic part of strain increment

\[
\Delta \varepsilon_e(\omega) = -\frac{p_1(\omega) - p_0(\omega)}{K(\omega)} \mathbf{I} + \frac{1}{G(\omega)}(\sigma_1^D(\omega) - \sigma_0^D(\omega))
\]

• plastic part of strain increment

\[
\Delta \varepsilon_p(\omega) = \Delta \varepsilon(\omega) - \Delta \varepsilon_e(\omega)
\]
Stochastic Yield Criteria

- Yield condition

\[ \| \sigma_e^D (\omega) \|_{J_2} = \sqrt{\frac{3}{2}} \| \sigma_e - \frac{1}{3} \text{tr}(\sigma_e)I \| \]

- Finding root from PCE and get a new PCE
  - using Monte Carlo simulation
  - using Newton’s method
  - using Taylor series
  - using Laguerre or Čebyšev expansion
  - using Padé approximation
  - using combination of previous methods with Newton’s method
Plane strain problem

- Plane strain

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & 0 \\
\sigma_{xy} & \sigma_{yy} & 0 \\
0 & 0 & \sigma_{zz}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & 0 \\
\varepsilon_{xy} & \varepsilon_{yy} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

- von Mises plasticity is pressure independent, hence stress / strain coordinates orthogonal to pressure line.

- outputs—pressure: \( p = -\frac{1}{3} \text{tr}(\sigma) \), in-plane shears \( \sigma_{xy} \) and \( \sigma_\delta = \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \), and out-of-plane shear \( \sigma_\Delta = \frac{1}{4}(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy}) \)
Pressure as Output

\[ p_\sigma = -\frac{1}{3} \text{Tr}(\sigma) \quad \mu_p = -1.7227 \quad \sigma_p = 1.3206 \]
Stress as Output

\[ \sigma_\Delta = \frac{1}{6} \left( -\left( \sigma_{xx} + \sigma_{yy} \right) + 2\sigma_{zz} \right) \]

\[ \mu = 0.3578, \ \sigma = 0.3684 \]
Strain as Output

\[
\begin{align*}
\mu &= -0.1589 \\
\sigma &= 2.1002 \\
P(\varepsilon_{xy} \geq 0.05) &= 0.4604
\end{align*}
\]
Summary

- Stochastic plasticity can be formulated in white noise setting.
- Stochastic Galerkin / collocation methods work for elasticity.
- They are computationally possible on today's hardware.
- They can be extended to plasticity (or other irreversible materials).